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IMPROVED CONTINUITY CRITERIA FOR LOCALLY BOUNDED HOMOMORPHISMS BETWEEN CENTRAL EXTENSIONS OF PERFECT LIE GROUPS

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Abstract.

We prove that every locally bounded homomorphism, between connected Lie central extensions of connected perfect Lie groups, where the source group is linear, taking the extending connected Abelian normal subgroup of the source group into the extending connected Abelian normal subgroup of the target group is continuous if and only if it is continuous on the extending connected Abelian normal subgroup of the source group.

§ 1. Introduction

As was proved in [1], every locally bounded homomorphism, between linear connected Lie central extensions of connected perfect Lie groups, taking the center of the source group into the center of the target group is continuous if and only if it is continuous on the center of the source group. In this paper, extending this assertion and applying the result of [2], we omit the linearity assumption and prove that every locally bounded homomorphism, between connected Lie central extensions of connected perfect Lie groups, where the source group is linear, taking the extending connected Abelian normal subgroup of the source group into the extending connected Abelian

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX normal subgroup of the target group is continuous if and only if it is continuous on the extending connected Abelian normal subgroup of the source group.

§ 2. Preliminaries

Let us recall some information needed below.

A (not necessarily continuous) homomorphism π of a topological group G into a topological group H is said to be *relatively compact* if there is a neighborhood $U = U_{e_G}$ of the identity element e_G in G whose image $\pi(U)$ has compact closure in H. Obviously, a homomorphism into a locally compact group is relatively compact if and only if it is *locally bounded*, i.e., there is a neighborhood U_e whose image is contained in some element of the filter $\mathfrak{V}$ of neighborhoods of e_V having compact closure.

Let us also recall the notion of discontinuity group of a homomorphism π of a topological group G into a topological group H, see [5] and [6]. Let $\mathfrak{U} = \mathfrak{U}_G$ be the filter of neighborhoods of e_G in G. For every (not necessarily continuous) locally relatively compact homomorphism π of G into H, the set

$$\mathrm{DG}(\pi) = \bigcap_{U \in \mathfrak{U}} \overline{\pi(U)}$$

is called the discontinuity group of π. Here and below, the bar stands for the closure in the corresponding topology (here the closure is taken in the topology of H). (See Definition 1.1.1 of [5].)

The discontinuity group of a homomorphism has some important properties. Under the above conditions, the set $\mathrm{DG}(\pi)$ is a compact subgroup of the topological group H and a compact normal subgroup of the closed subgroup $\overline{\pi(G)}$ of H. Moreover, the filter basis $\{\overline{\pi(U)} \mid U \in \mathfrak{U}\}$ converges to $\mathrm{DG}(\pi)$, and the homomorphism π is continuous if and only if $\mathrm{DG}(\pi) = \{e_H\}$. (See Theorem 1.1.2 of [5].) If G is a connected Lie group, then $\mathrm{DG}(\pi)$ is a compact connected subgroup of H. (See Lemma 1.1.6 of [5].) Let G be a connected Lie group, let N be a closed normal subgroup of G, and let π be a locally bounded homomorphism of G into a locally compact group H. Let M be the discontinuity group of the restriction $\mathrm{DG}(\pi|_N)$. Then M is a closed normal subgroup of the compact discontinuity group $\mathrm{DG}(\pi)$, and the corresponding quotient group $\mathrm{DG}(\pi)/M$ is isomorphic to the discontinuity group $\mathrm{DG}(\psi)$ of the homomorphism ψ of G obtained as the composition of the homomorphism π and the canonical homomorphism $\overline{\pi(G)} \to \overline{\pi(G)}/M$. (See Lemma 1.1.7 of [5].)

§ 3. Main result

Theorem 1. *Every locally bounded homomorphism, of a connected linear Lie central extension $G = G_1$ of a connected perfect Lie group H_1 by the commutative connected Lie group Z_1 into a connected Lie central extension G_2 of a connected perfect Lie group H_2 by the commutative connected Lie group Z_2, taking Z_1 into Z_2, is continuous if and only if it is continuous on Z_1.*

Proof. Obviously, if an endomorphism of a group is continuous, then it is continuous on every subgroup of the group of the group, and thus it suffices to prove the "if" part of the assertion. Let H be a perfect Lie group, let Z be an Abelian Lie group, and let a connected Lie group G enter a short exact sequence $\{e\} \to Z \xrightarrow{\iota} G \xrightarrow{\rho} H \to \{e\}$ with a continuous embedding ι and the canonical epimorphism ρ of G onto H, which is isomorphic to the quotient group G/Z. Then the commutator subgroup G' of G is closed (see Chap. 3, Exercise 41(e) of [2]) and is taken by ρ onto the commutator subgroup DH of H. Moreover, G' is in a natural one-to-one correspondence with DH. Indeed, for every $z_1, z_2 \in Z$ and $b, c \in G$, we have $bz_1c_2(bz_1)^{-1}(cz_2)^{-1} = bzb^{-1}c^{-1} = [b, c]$ and, therefore, the commutator of bZ and cZ is $[b, c]Z$ for every $b, c \in G$. Thus, the commutator subgroup of G is naturally isomorphic to that of H. However, H is perfect, and hence $DH = H$. Hence the closed subgroup G' of G is naturally isomorphic to H, and every element of G is a product of an element of G' and an element of Z. Thus, the extension in question is split.

Let π be a locally bounded homomorphism of $G = G_1$ into a connected Lie central extension G_2 of a connected perfect Lie group H_2 by the commutative connected Lie group Z_2, taking Z_1 into Z_2. It naturally takes the commutator subgroup G' into the commutator subgroup of G_2. By the very assumption concerning π in the "if" part of the theorem, the restriction of π to Z is continuous. By the theorem in [2], the restriction of π to G' is continuous (cf. [3] and [4]). Hence the representation π is separately continuous with respect to the subgroups Z and G'.

By the Namioka theorem [6], the representation π has a point of joint continuity, and therefore is continuous. This completes the proof of Theorem 1.

§ 4. Comments

The phenomenon relating the continuity of a locally bounded endomorphism to the continuity of its restriction to the center is specific, see the example in [1].

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